Lesson 12. Linear Systems and Matrices

0 Warm up

Example 1. Solve for x: $ax = b$ ($a \neq 0$)

1 Systems of linear equations using matrices

- We can write the following system of linear equations

$$
\begin{aligned}
6x_1 + 3x_2 + x_3 &= 22 \\
x_1 + 4x_2 - 2x_3 &= 12 \\
4x_1 - x_2 + 5x_3 &= 10
\end{aligned}
$$

  using matrices and vectors:

- It would be nice if we could write:

$$AX = b \quad \Leftrightarrow \quad A^{-1}AX = A^{-1}b \quad \Leftrightarrow \quad X = A^{-1}b \qquad (\star)$$

2 The inverse of a matrix

- Let A be a $n \times n$ (square) matrix
- The **inverse** of a matrix A is denoted by A^{-1}
  - A^{-1} is also $n \times n$
  - A^{-1} satisfies

Example 2. Let

$$A = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 6 \\ 1 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Is B the inverse of A? Is C the inverse of A?

- Properties of inverses:
  - A^{-1} is defined only if A is square
  - A^{-1} does not necessarily exist
    - A is **nonsingular** if it has an inverse (a.k.a. **invertible**)
    - A is **singular** if it has no inverse
  - If A^{-1} exists, then A^{-1} is unique
  - $AA^{-1} = I$ implies $A^{-1}A = I$ and vice-versa
  - Suppose A and B are nonsingular and square. Then:
    - $(A^{-1})^{-1} = A$
    - $(AB)^{-1} = B^{-1}A^{-1}$
    - $(A^T)^{-1} = (A^{-1})^T$

Example 3. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} =$
- We'll learn how to find the inverse of any sized (square) matrix soon

Example 4. Solve the system below using inverses:

$$\begin{aligned} 2x - 5y &= 1 \\ -5x + 3y &= 3 \end{aligned}$$

Hint. Use ($\star$) on page 1. To save some time, look at Example 2.

3 Another warm up

Example 5. Rewrite the system of linear equations

$$\begin{aligned} 2x + 4z &= 2 - 8y \\ z &= 5 - 2x - 5y \\ 4x + 10y - z &= 1 \end{aligned} \tag{A}$$

using matrices, assuming the three variables are arranged in the order x, y, z.

4 The augmented matrix

- The matrix $\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}$ is the **coefficient matrix** for the system (A)
    - Columns of the coefficient matrix $\Leftrightarrow$ Variables in the system
- The matrix $\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right]$ is the **augmented matrix** for the system (A)
- To solve (A), we will work with its augmented matrix

5 Reduced row echelon form (RREF)

- A matrix is in **row echelon form** if
  1. Nonzero rows appear above the zero rows
  2. In any nonzero row, the first nonzero entry is a 1 (this is called the **leading one**)
  3. The leading one in a nonzero row appears to the left of the leading one in any lower row
- A matrix is in **reduced row echelon form (RREF)** if conditions 1-3 above are satisfied and
  4. If a column contains a leading one, then all the other entries in that column are 0

Example 6. Are the following matrices in row echelon form? RREF?

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

- Why is RREF useful?
- Take a look at the 4th matrix in Example 6
- Suppose it is the augmented matrix of a system of equations with variables x, y, z
- The system is

⇒ If we can transform augmented matrices into RREF in a systematic way, then we can solve systems of linear equations in a systematic way

- What are these "transformations"?

6 Elementary row operations

- The **elementary row operations** are
  1. Divide a row by a nonzero number
  2. Subtract a multiple of a row from another row
  3. Interchange two rows
- Performing an elementary row operation on an augmented matrix
  - ⇒ Get new augmented matrix representing new system of equations AND new system of equations has the same solutions as the original

- Strategy to transform augmented matrix into RREF:
  - First, transform entries in the lower left into 0s $\Longrightarrow$ row echelon form
  - Second, transform entries above the leading ones into 0s $\Longrightarrow$ RREF

Example 7. Transform the augmented matrix $\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$ of system (A) into RREF.

7 Finding solutions from RREF

- Once an augmented matrix is transformed into RREF, how do we find the solutions of the corresponding system of linear equations?
- A column that contains a leading one is called a **leading column**
- A variable that corresponds to a leading column is called a **leading variable**
- The non-leading variables are called **free variables**
- To find all solutions to the system, solve the equations for the leading variables in terms of the free variables

Example 8. In the RREF matrix you found in Example 7, what are the leading variables? What are the free variables? Use the RREF matrix to solve the system (A).

Example 9. Suppose the RREF of the augmented matrix of a system of linear equations is

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the corresponding system? What are the leading variables? What are the free variables? What are the solutions of this system?

Example 10. Suppose the RREF of the augmented matrix of a system of linear equations is

$$\begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the corresponding system? What are the leading variables? What are the free variables? What are the solutions of this system?

- Possible outcomes for a system of linear equations:

1.
2.
3.

Example 11. Consider the following system of equations:

$$\begin{aligned} x + y - 2z &= 1 \\ -x + 10z &= -1 \\ 2x + 3y + 4z &= 2 \end{aligned}$$

a. Form the augmented matrix for this system.
b. Solve this system by putting its augmented matrix into RREF.

8 Finding the inverse of a matrix with RREF

- To find the inverse of an $n \times n$ matrix A, compute the RREF of $\begin{bmatrix} A & I_n \end{bmatrix}$
  - If the RREF of $\begin{bmatrix} A & I_n \end{bmatrix}$ has the form $\begin{bmatrix} I_n & B \end{bmatrix}$, then A is invertible and $A^{-1} = B$
  - If the RREF of $\begin{bmatrix} A & I_n \end{bmatrix}$ has another form (i.e., the left side fails to be I_n), then A is not invertible

Example 12. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.